7.5

Midtoets Vectoranalyse 25-5-'10

(1) S: $x^2+2y^2+3z^2=6$

raakvlak in (a,b,c): $2a(x-a)+4b(y-b)+6c(y-c)=0$

(6,0,0): $2a(6-a)+4b(0-b)+6c(0-c)=0$

$12a-2a^2-4b^2-6c^2=0$

(0,0,2): $2a(0-a)+4b(0-b)+6c(2-c)=0$

$-2a^2-4b^2+12c-6c^2=0$

$\Rightarrow 12a-\cancel{2a^2}-4b^2-\cancel{6c^2}=-\cancel{2a^2}-4b^2+12c-\cancel{6c^2}$

$\Rightarrow a=c$

$12a-8a^2=4b^2$

$3a-2a^2=b^2$

$b=\pm\sqrt{a(3-2a)}$ 1?

$(a,b,c)=k(1,\pm\sqrt{3},1)$ ~ reken fout?

invullen: $k^2+4\cdot3k^2+3k^2=6$???

$16k^2=6$

$k^2=\frac{3}{8}$ $\qquad \sqrt{\frac{3}{8}}=\frac{\sqrt{3}}{2\sqrt{2}}=\frac{\sqrt{3}\sqrt{2}}{4}=\frac{\sqrt{6}}{4}$

$k=\pm\sqrt{\frac{3}{8}}$ $\qquad \sqrt{\frac{3}{8}}\cdot\sqrt{3}=\sqrt{\frac{9}{8}}=\frac{3}{\sqrt{8}}=\frac{3}{2\sqrt{2}}=\frac{3}{4}\sqrt{2}$

punten zijn: $(\frac{1}{4}\sqrt{6}, \frac{3}{4}\sqrt{2}, \frac{1}{4}\sqrt{6})$ en

$(\frac{1}{4}\sqrt{6}, -\frac{3}{4}\sqrt{2}, \frac{1}{4}\sqrt{6})$ en

$(-\frac{1}{4}\sqrt{6}, -\frac{3}{4}\sqrt{2}, -\frac{1}{4}\sqrt{6})$ en

$(-\frac{1}{4}\sqrt{6}, \frac{3}{4}\sqrt{2}, -\frac{1}{4}\sqrt{6})$

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(2) i. $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$

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$\frac{\partial u}{\partial x}=e^{x}$ $\quad \frac{\partial v}{\partial x}=-e^{-x}$ $\quad \frac{\partial u}{\partial y}=e^{y}$ $\quad \frac{\partial v}{\partial y}=-e^{-y}$

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$=\frac{\partial z}{\partial u}\left(e^{x}+e^{y}\right)+\frac{\partial z}{\partial v}\left(-e^{-x}-e^{-y}\right)$$

$$=u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v} \qquad \square$$

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2. 2.0.2.

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2.) $\cancel{\frac{\partial^2 z}{\partial u^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 z}{\partial u \partial v}} \frac{\partial^2 z}{\partial v^2}\frac{\partial^2 v}{\partial x^2}$

$+ \frac{\partial^2 z}{\partial u^2}\frac{du}{dx}\frac{du}{dx} + \frac{\partial^2 z}{\partial v^2}\frac{dv}{dx}\frac{dv}{dx} + 2\frac{du}{dx}\frac{dv}{dx}\frac{d^2 z}{\partial u \partial v}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right) = \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2} = \frac{\partial z}{\partial u}e^x + \frac{\partial z}{\partial v}e^{-x}$$

$+ \ldots$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right) = \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2} = \frac{\partial z}{\partial u}e^y + \frac{\partial z}{\partial v}e^{-y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) + \ldots$$

$$\frac{\partial}{\partial y}\frac{\partial z}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}e^x - \frac{\partial z}{\partial v}e^{-x}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}e^x\right) = \frac{\partial^2 z}{\partial u \partial y}e^x = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}e^x\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)e^x$$

$$= \left(\frac{\partial^2 z}{\partial u^2}e^y - \frac{\partial^2 z}{\partial u \partial v}e^{-y}\right)e^x$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}e^{-x}\right) = \frac{\partial^2 z}{\partial v \partial y}e^{-x} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}e^{-x}\right) = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)e^{-x}$$

$$= \left(\frac{\partial^2 z}{\partial u \partial v}e^y - \frac{\partial^2 z}{\partial v^2}e^{-y}\right)e^{-x}$$

$$\Rightarrow \frac{\partial}{\partial y}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial u^2}e^{x+y} - \frac{\partial^2 z}{\partial u \partial v}e^{x-y} - \frac{\partial^2 z}{\partial u \partial v}e^{-x+y} + \frac{\partial^2 z}{\partial v^2}e^{-x-y}$$

$$\frac{\partial}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}e^y - \frac{\partial z}{\partial v}e^{-y}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}e^y\right) = \frac{\partial^2 z}{\partial u \partial x}e^y = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)e^y = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)e^y$$

$$= \left(\frac{\partial^2 z}{\partial u^2}e^x - \frac{\partial^2 z}{\partial u \partial v}e^{-x}\right)e^y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}e^{-y}\right) = \frac{\partial^2 z}{\partial v \partial x}e^{-y} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)e^{-y} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)e^{-y}$$

$$= \left(\frac{\partial^2 z}{\partial u \partial v}e^x - \frac{\partial^2 z}{\partial v^2}e^{-x}\right)e^{-y}$$

$$\Rightarrow \frac{\partial}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial u^2}e^{x+y} - \frac{\partial^2 z}{\partial u \partial v}e^{-x+y} - \frac{\partial^2 z}{\partial u \partial v}e^{x-y} + \frac{\partial^2 z}{\partial v^2}e^{-x-y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2}e^{x+y} - \frac{\partial^2 z}{\partial u \partial v}e^{-x+y} - \frac{\partial^2 z}{\partial u \partial v}e^{x-y} + \frac{\partial^2 z}{\partial v^2}e^{-x-y}$$

$$u^2 = e^{2x} + 2e^{x+y} + e^{2y}$$

$$uv = 1 + e^{x-y} + e^{-x+y} + 1$$

$$v^2 = e^{-2x} + 2e^{-x-y} + e^{-2y}$$

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$$2\frac{\partial^2 z}{\partial x \partial y} = 2\frac{\partial^2 z}{\partial u^2} e^{x+y} - 2\frac{\partial^2 z}{\partial u \partial v} e^{-x+y} - 2\frac{\partial^2 z}{\partial u \partial v} e^{x-y} + \frac{\partial^2 z}{\partial v^2} e^{-x-y}$$

$$= \underbrace{\left(u^2 - e^{2x} - e^{2y}\right)\frac{\partial^2 z}{\partial u^2} - 2(uv-2)\frac{\partial^2 z}{\partial u \partial v} + \left(v^2 - e^{-2x} - e^{-2y}\right)\frac{\partial^2 z}{\partial v^2}}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{2\partial^2 z}{\partial x \partial v} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial u}\left(e^x + e^y\right) + \frac{\partial z}{\partial v}\left(e^{-x} + e^{-y}\right) +$$

$$= \left(u^2 - e^{2x} - e^{2y}\right)\frac{\partial^2 z}{\partial u^2} - 2(uv-2)\frac{\partial^2 z}{\partial u \partial v} + \left(v^2 - e^{-2x} - e^{-2y}\right)\frac{\partial^2 z}{\partial v^2} + u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$$

dit klopt bijna. ergens is wat weggevallen, maar ik weet niet waar.

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3) $f(x,y,z) = 2x - y - z$

$g_1(x) = x + y + z = 0$

$g_2(x) = x^2 + y^2 + z^2 - 1 = 0$

extrema: $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$

$$\begin{cases} 2 = \lambda_1 + 2x\lambda_2 & (1) \\ -1 = \lambda_1 + 2y\lambda_2 & (2) \\ -1 = \lambda_1 + 2z\lambda_2 & (3) \\ x+y+z=0 & (4) \\ x^2+y^2+z^2=1 & (5) \end{cases}$$

uit (2) en (3) $\Rightarrow y = z$

$x = \frac{2-\lambda_1}{2\lambda_2}$

$y = z = \frac{-1-\lambda_1}{2\lambda_2}$

$x+y+z=0$

$\Rightarrow \frac{2-\lambda_1 - 1 - \lambda_1 - 1 - \lambda_1}{2\lambda_2} = 0$

$\Rightarrow -3\lambda_1 = 0$

$\lambda_1 = 0$

$\Rightarrow x = \frac{1}{\lambda_2}$

$y = z = \frac{-1}{2\lambda_2}$

$x^2+y^2+z^2=1$

$\frac{1}{\lambda_2^2} + \frac{1}{4\lambda_2^2} + \frac{1}{4\lambda_2^2} = \frac{6}{4\lambda_2^2} = 1$

$\Rightarrow 6 = 4\lambda_2^2$

$\lambda_2^2 = \frac{3}{2}$ $\qquad (x,y,z) = \left(\frac{1}{\sqrt{\frac{3}{2}}}, \frac{-1}{2\sqrt{\frac{3}{2}}}, \frac{-1}{2\sqrt{\frac{3}{2}}}\right) := P_1$

$\lambda_2 = \pm\sqrt{\frac{3}{2}}$ $\qquad$ of $(x,y,z) = \left(\frac{-1}{\sqrt{\frac{3}{2}}}, \frac{1}{2\sqrt{\frac{3}{2}}}, \frac{1}{2\sqrt{\frac{3}{2}}}\right) := P_2$

~~Maxtab~~

$P_1 = \left(\sqrt{\tfrac{2}{3}}, \cancel{\ \ } -\sqrt{\tfrac{1}{6}}, -\sqrt{\tfrac{1}{6}}\right)$

$P_2 = \left(-\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{6}}, \sqrt{\tfrac{1}{6}}\right)$

$f(P_1) = 2\sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{1}{6}} + \sqrt{\tfrac{1}{6}} = 2\sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{2}{3}} = 3\sqrt{\tfrac{2}{3}}$

$f(P_2) = -2\sqrt{\tfrac{2}{3}} - \sqrt{\tfrac{1}{6}} - \sqrt{\tfrac{1}{6}} = -2\sqrt{\tfrac{2}{3}} - \sqrt{\tfrac{2}{3}} = -3\sqrt{\tfrac{2}{3}}$

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b) $f(x,y,z)$ is op de cirkel besloten en begrensd en f is continu
dus zegt Weierstrass dat f op de cirkel een minimum en een maximum
moet hebben, dit moeten P_1 en P_2 zijn
minimum: $\left(-\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{6}}, \sqrt{\tfrac{1}{6}}\right) = P_2$
maximum: $\left(\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{6}}, \sqrt{\tfrac{1}{6}}\right) = P_1$